RADIALLY SYMMETRIC FLOW OF A VISCOUS LIQUID BETWEEN PARALLEL PLANES

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Axisymmetric spreading of a viscous liquid between parallel planes is studied under conditions of frontal polymerization in the flow. Conditions required for existence of steady-state regimes are determined, together with their number, properties, and stability. Analytical expressions are obtained for the critical parameter values at which steady-state regimes disappear or lose stability. An estimate of maximum reactor output is also obtained.

1. Introduction. Frontal polymerization within a flow is one of the most promising methods of polymer production, although the possibilities of realizing this process in tube reactors are limited due to adhesion of viscous reaction products to the tube walls, development of jet flows, and reactor breakdowns [1]. Therefore, it is of interest to study reactors of other types, in particular, cylindrical and spherical ones, in which one can expect wall effects to be less significant. Cylindrical reactors have been examined in a number of studies [2-5], although hydrodynamic effects were not considered. The present study will demonstrate that consideration of hydrodynamic effects limits the parameter range over which stable steady-state regimes can be realized.

2. Formulation of Problem. We will study axisymmetric flow of a viscous liquid in a cylindrical reactor with attachment conditions on the faces. The cylindrical reactor model which will be considered consists of two coaxial cylinders bounded by parallel planes (the reactor end plates). With internal supply liquid is fed into the reactor through the inner cylinder (radius r_0) and removed through the outer one (radius r_1). Correspondingly, with external supply liquid is fed through the outer cylinder and extracted through the inner one. The subscripts 0 and 1 below will refer to values on the inner and outer surfaces.

The solution of the equations of motion of the viscous liquid and the continuity equation, written in cylindrical coordinates r, φ, z (where z is the coordinate along the cylinder axis) will be sought in the form $V_r = V_r(r, z, t)$, $V_\varphi = 0$, $\partial V_r/\partial\varphi = 0$, with the assumption $V_z \ll V_r$, i.e., a flow close to radial will be considered. This is physically justifiable in the case where the velocity component V_z is equal to zero on the inner and outer reactor surfaces, which corresponded to experimental conditions. The validity of the assumption $V_z \ll V_r$ can easily be demonstrated for the case of low flow velocities, where the quadratic terms of the motion equation may be neglected, and for high velocities, where the quadratic terms become dominant and the solution depends only weakly on the pressure head over the radius. We will note that smallness of V_z permits us to assume that the pressure is practically independent of z, while the pressure head between the inner and outer surfaces is a constant, which will be assumed known (as is realized in practice).

With these assumptions, after integrating the continuity equation over r we obtain

$$V_r = \frac{1}{r} f(z,\ t). \tag{1}$$

We will solve the equation of motion for the velocity component V_r, using Eq. (1):

$$\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + V_z\frac{\partial V_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_r}{\partial r}\right) + \frac{\partial^2 V_r}{\partial z^2} - \frac{V_r}{r^2}\right]. \tag{2}$$

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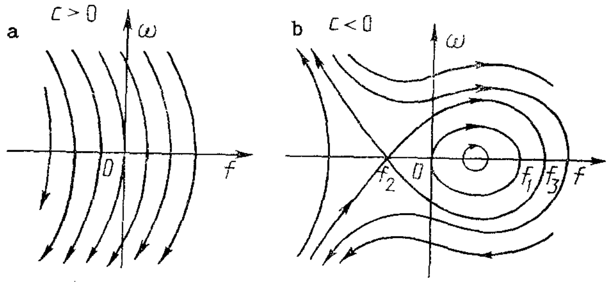

Fig. 1. Phase plane trajectories of system (6); $a > 0$, $b > 0$.

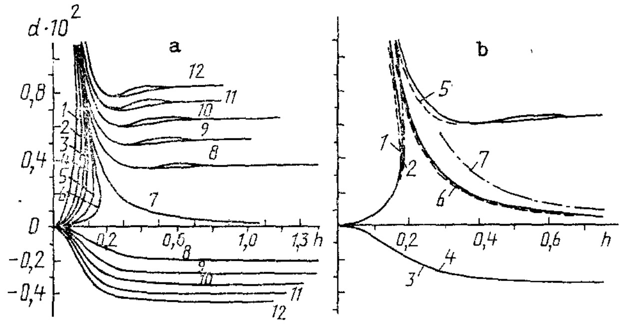

Fig. 2. Maximum flow velocity vs reactor semi-height (a) and values thereof (b) for various c ($a = 46.05 \cdot 10^{-4}$ m²/sec, $b = 4999.5$ m^{-2}): a) 1 – $c = 20 \cdot 10^{-6}$ m²/sec²; 2 – $10 \cdot 10^{-6}$; 3 – $6 \cdot 10^{-6}$; 4 – $4 \cdot 10^{-6}$; 5 – $2 \cdot 10^{-6}$; 6 – 10^{-6}; 7 – 0; 8 – $-2 \cdot 10^{-6}$; 9 – $-4 \cdot 10^{-6}$; 10 – $-6 \cdot 10^{-6}$; 11 – $8 \cdot 10^{-6}$; 12 – $-10 \cdot 10^{-6}$; b) 1, 2 – $c = 6 \cdot 10^{-6}$; 1 – $h_1^-(d)$; 2 – $h_1^+(d)$; 3, 4, 5 – $c = -6 \cdot 10^{-6}$; 3 – $h_2^-(d)$; 4 – $h_2^+(d)$; 5 – $h_1^-(d)$; 6, 7 – $c = 0$; 6 – $h_1^-(d)$; 7 – $h_1^+(d)$. d, m²/sec; h, m.

Here $r_0 \le r \le r_1$, $0 \le z \le 2h$. We will consider the boundary conditions $V_r|_{z=0} = V_r|_{z=2h} = 0$. For solutions symmetric about the plane $z = h$, the boundary conditions will be specified in the form $V_r|_{z=0} = 0$, $\partial V_r/\partial z|_{z=h} = 0$.

After integrating Eq. (2) over r from r_0 to r_1 and substitution in Eq. (1), dropping small quantities, we obtain an equation for the function $f(z, t)$:

$$f_t = \frac{1}{\ln \frac{r_1}{r_0}} (af_{zz} + bf^2 + c). \tag{3}$$

Here we use the notation: $a = \nu \ln \frac{r_1}{r_0}$, $b = \frac{1}{2}\left(\frac{1}{r_0^2} - \frac{1}{r_1^2}\right)$, $c = -\Delta p/\rho$, $\Delta p = p_1 - p_0$ is the known pressure head, $f_t = \partial f/\partial t$, $f_{zz} = \partial^2 f/\partial z^2$.

3. Steady-State Case. In the steady state the function f satisfies the equation

$$af_{zz} + bf^2 + c = 0 \tag{4}$$

on the section $0 \le z \le 2h$ and boundary conditions

$$f(0) = f(2h) = 0. \tag{5}$$

Equation (4) can be solved in quadratures, although it will be convenient to first find the qualitative characteristics of the solutions. To do this we reduce the equation to a system of two first order equations

$$f_z = w,\ w_z = -\frac{1}{a}(bf^2 + c). \tag{6}$$

Figure 1 shows the qualitative behavior of trajectories in (6). The solutions of Eq. (4) with boundary conditions (5) obviously correspond to trajectory arcs beginning and ending on the ordinate axis. Therefore, in the case $c > 0$ (Fig. 1a) $f(z) \geq 0$ for all z, i.e., outward flow is always realized. In this case $f(z)$ is symmetric about $z = h$.

For the case $c < 0$ (Fig. 1b) the solutions may be positive (for $0 < z < 2h$), negative, or sign-changing. If for the parameters we specify the values $f^+ = \max\limits_{0<z<2h} f(z)$ and $f^- \min\limits_{0<z<2h} f(z)$ then, as can easily be seen, independent of the reactor height position solutions (i.e., outward flow) exist at $f^+ \geq f_1$, negative ones (i.e., flow inward) at $f^- > f_2 = -\sqrt{-c/b}$, and sign-changing ones at $f_1 \leq f^+ < f_3$, $f^- > f_2$. The positive and negative solutions are symmetric about $z = h$, while the sign-changing ones may be either symmetric or antisymmetric.

We will note that for an infinite reactor $(-\infty \leq z \leq +\infty)$ in the case $c > 0$, as will be made clear below, finite steady-state solutions do not exist. For the case $c < 0$ such solutions may be of three types: constant over z ($f(z) \equiv \pm\sqrt{-c/b}$), periodic over z ($f_1 < f^+ < f_3$, $f^- > f_2$), and a sign-changing solution for which $f^+ = f_3$, $f(\pm\infty) = f_2$.

Neither is it difficult to find steady-state flow regimes in a semiinfinite reactor $(0 \leq z \leq +\infty)$.

It can be shown that the constant sign solution of Eq. (4) with conditions $f(0) = 0$, $f_z(h) = 0$ can be specified in the form

$$\int\limits_0^{f(z)} \frac{d\tau}{\sqrt{-\frac{2}{a}\left[\frac{b}{3}(\tau^3 - d^3) + c(\tau - d)\right]}} = z, \tag{7}$$

where $d = f(h)$ (if $d \neq 0$, then $d = f^{\pm}$). The sign-changing solutions can be "compiled" from constant sign solutions. The function $f(z)$ can be found from the implicit expression of Eq. (7) numerically, and explicit estimates obtained.

Figure 2a shows the dependence of d on h for various values of c, obtained by numerical solution of Eq. (7). These curves allow us to determine some principles describing the solution behavior. For the case $\Delta p < 0$ (curves 1-6) for each value of pressure head and quantity $h < h_{cr}^{(1)}$ two positive solutions of Eq. (4) exist, which correspond to different maximum velocity values. For $h > h_{cr}^{(1)}$ steady-state solutions do not exist, i.e., production of a given pressure head in a reactor of that height is impossible.

Two solutions of Eq. (4) also exist for $\Delta p = 0$ for each h, one of which is positive (curve 7), and other other, identically equal to zero.

For $\Delta p > 0$ positive, negative, and sign-changing solutions may exist. The negative solution (i.e., flow inward) exists for any reactor height and $d \to -\sqrt{-c/b}$ as $h \to \infty$ (curves 8-12 in region $d < 0$). Positive solutions (i.e., flow outward) exist only for $h < h_{cr}^{(2)}$ (the single-valued branches of curves 8-12 in the region $d > 0$). At $h = h_{cr}^{(2)}$ ($f^+ = f_1$) the positive solution transforms into three sign-changing solutions (weak degeneration), which exist for all $h > h_{cr}^{(2)}$. As $h \to \infty$ we have $f^+ \to f_3$, $f^- \to f_2$. Moreover, at $h = nh_{cr}^{(2)}$, n = 1, 2, ..., for each n value there appear another three-sign-changing solutions (strong degeneration), for which the intervals of positive and negative values of the function $f(z)$ alternate, their number increasing with increase in n (number of positive intervals equal to n). We note here some analogy to the flow of a viscous liquid in a diffusor [6].

As was noted above, the representation of Eq. (7) can be used to obtain concrete estimates. To do this, it is obviously necessary only to evaluate the integrand, since the integral of Eq. (7) can be reduced to tabular form. In particular, for $\Delta p < 0$ we have the following estimate of the quantity h as a function of d (Fig. 2b, curves 1, 2):

$$h_1^-(d) < h < h_1^+(d),$$

where

$$h_1^-(d) = \sqrt{\frac{3a}{4bd}} \arccos \frac{3c - bd^2}{3(c + bd^2)},$$

$$h_1^+(d) = \sqrt{\frac{a}{2bd}} \arccos \frac{c - bd^2}{c + bd^2}.$$

For $\Delta p > 0$ for inward flows ($d < 0$)

$$h_2^-(d) < h < h_2^+(d)$$

(Fig. 2b, curves 3, 4), where

$$h_2^-(d) = -\sqrt{\frac{-3a}{4bd}} \ln \frac{3(c + bd^2)}{3c - bd^2 + 2d\sqrt{-2b(3c + bd^2)}},$$

$$h_2^+(d) = -\sqrt{\frac{-a}{2bd}} \ln \frac{c + bd^2}{c - bd^2 + 2d\sqrt{-bc}}.$$

For $\Delta p > 0$ for outward flow ($d > 0$) $h > h_1^-(d)$ (Fig. 2b, curve 5). We note that the functions $h_1^-(d)$ and $h_2^-(d)$ give a quite precise estimate and can be used as approximate values.

It can easily be proved that at $d \gg \sqrt{c/b}$

$$h_1^+(d) \sim \pi\sqrt{\frac{a}{2bd}},$$

while at $d \ll \sqrt{c/b}$

$$h_1^+(d) \sim \sqrt{\frac{2ad}{c}}, \tag{8}$$

whence we can obtain the approximate value of $h_{cr}^{(1)}$

$$h_{cr}^{(1)} \approx \sqrt{\frac{\pi a}{\sqrt{bc}}}$$

corresponding to the d value:

$$d_{cr}^{(1)} \approx \frac{\pi}{2}\sqrt{\frac{c}{b}}.$$

We recall that for $\Delta p < 0$ flow outward exists only at $h < h_{cr}^{(1)}$, which imposes definite limitations on reactor geometric characteristics. We will also note that of the two branches of the d(h) curve the stable solutions correspond to the lower branch (see Sec. 3), so that independent of reactor height the maximum stable flow velocity is always bounded above ($d \leq d_{cr}^{(1)}$). The quantity d may then become comparable to $d_{cr}^{(1)}$ even for relatively slow viscous liquid flows, so that the asymptote of Eq. (8), (or, what is the same, neglect of quadratic terms in the motion equations) may not always be used.

4. Nonsteady Case. If the initial velocity distribution is independent of angle and inversely proportional to radius, i.e., $V_r^0 = (1/r)f^0(z)$ ($V_\varphi^0 = V_z^0 = 0$), this will also be true for the velocity distribution at any point in time. This means that the function V_r may be written in the form of Eq. (1), where f(z, t) is a solution of Eq. (3) with boundary conditions (5) and initial condition $f(z, 0) = f^0(z)$. Methods for study of the solutions of such boundary problems are well developed and are based on the positiveness theorems and comparison theorems for linear and quasilinear parabolic equations.

We will note that the steady-state solutions of the boundary problem under consideration which are not identically equal to zero are obviously nonmonotonic. It is known (see [7]) that such solutions are unstable, if they possess two or more extrema. If there is but one extremum then the steady-state solution may be either stable or unstable. Simple analysis of the behavior of the solutions of Eq. (3) using the comparison theorem shows that in the case $\Delta p < 0$ the steady-state solutions corresponding to the upper branch of d(h) (Fig. 2a) are unstable, while those corresponding to the lower branche are stable. Similarly, for the case $\Delta p > 0$ those solutions with negative d are stable (flow inward), while those with positive d (flow outward) and those which change sign are unstable.

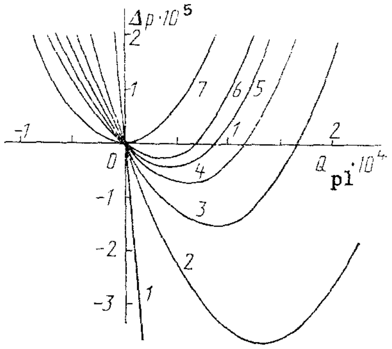

Fig. 3. Reactor pressure head-output characteristics for various h values: 1) h = 0.1 m; 2) 0.2; 3) 0.25; 4) 0.3; 5) 0.35; 6) 0.4; 7) ∞; $a = 46.05 \cdot 10^{-4}$ m^2/sec, $b = 4999.5\ m^{-2}$; $\rho = 10^3\ kg/m^3$, Δp, Pa; Q_{pl}, m^2/sec.

These conclusions were confirmed by direct numerical solution of the boundary problem. If for the initial condition a disturbed unstable steady-state solution was chosen, then for negative disturbances a stable steady state was established rapidly, while for positive disturbances the solution increased without limit over a finite time, corresponding to unlimited growth in flow velocity. However, it should be understood that at sufficient high flow velocities the formulation used herein becomes incorrect (in particular, one cannot specify Δp as a parameter). This is also true of the case $\Delta p < 0$, $h > h_{cr}^{(1)}$, where a steady-state solution of the form considered does not exist, and the solution of the nonsteady problem increases without limit for any initial condition.

To conclude this facet of the problem, we will briefly consider the question of stability of steady-state solutions in the case of an infinite reactor ($-\infty \le z \le +\infty$) for $\Delta p > 0$. In this case the only stable solution is $f(z) \equiv f_2$, while all remaining steady-state solutions are unstable. Of definite interest here is the process of transition of the solution from constant positive to constant negative; this transition is accomplished due to the propagation of two waves moving in opposite directions with velocity $2\sqrt{2a\sqrt{-bc}}$ (see [8, 9]).

5. Calculation of Reactor Output. The volume output of the mixture through the cylindrical surface r = const can be expressed in terms of the function f(z) as

$$Q = 2\pi \int_0^{2h} f(z)\, dz. \tag{9}$$

We will choose as the characteristic of reactor productivity the weight-averaged "planar output" $Q_{pl} = Q/2h$. Then, using Eq. (9) and the numerical solution of Eq. (7), we can construct reactor pressure head-output characteristics (see Fig. 3). In the case of a reactor of infinite height, the head-output characteristic has the form

$$\Delta p = \frac{\rho b}{4\pi^2} Q_{pl}^2,$$

while $|f(z)| = \sqrt{|c|/b}$. In a reactor of finite height, all steady-state flows directed inward are stable. In this case $\Delta p > 0$ and $|f(h)| < \sqrt{-c/b}$. We will assume that the height 2h is sufficiently large that everywhere but in a narrow boundary layer we may take $f(z) = -\sqrt{-c/b}$, then

$$Q_{pl} \simeq -2\pi \sqrt{-c/b}.$$

If in addition $r_1 \gg r_0$, then $b \approx 1/2r_0^2$ and

$$Q_{pl} \simeq -2\pi r_0 \sqrt{-2\Delta p/\rho}.$$

This approximation is valid for $h \gg h_{cr}^{(3)} = \sqrt{2a/\sqrt{-bc}}$. For $f(h) > 0$ solutions with $\Delta p < 0$ and $f(h) < d_{cr}^{(1)} \leqslant \frac{\pi}{2}\sqrt{\frac{c}{b}}$, are stable, i.e., those described by the quadratic expressions:

$$f(h) = \frac{c}{2a} h^2 \quad \text{and} \quad f(z) = -\frac{c}{2a} z^2 + \frac{ch}{a} z. \tag{10}$$

In this case Q_{pl} is defined as

$$Q_{pl} \simeq - \frac{\pi \Delta p}{3\rho\nu \ln \frac{r_1}{r_0}} h^2,$$

and the volume flow rate

$$Q \simeq - \frac{2}{3} \frac{\pi \Delta p}{\rho\nu \ln \frac{r_1}{r_0}} h^3.$$

Since h is is bounded above by the value $h_{cr}^{(1)} = \sqrt{\pi a/\sqrt{cb}}$, the maximum volume flow rate of mixture which can be realized in the reactor given the condition that the parameters a, b, c are specified and the flow is stable, will be:

$$Q_{max} \simeq \frac{2}{3} \frac{\pi^{2,5} c^{0,25} a^{0,5}}{b^{0,25}}.$$

6. Conclusion. Thus, steady-state flow regimes of a viscous liquid in a cylindrical reactor and their stability have been studied by approximate analytical and numerical methods. The dependence of maximum medium motion velocity upon reactor dimensions and pressure head-output characteristics have been constructed. Estimates of maximum reactor output have been presented. The results obtained permit determination of optimum process characteristics to accomplish frontal polymerization in cylindrical type reactors.

NOTATION

h, reactor half-height; p, pressure; Δp, pressure head between outer and inner reactor surfaces; Q, mixture volume flow rate; Q_{pl}, "planar" flow rate averaged over height; r, φ, z, cylindrical coordinates; r_0, r_1, radii of inner and outer cylinders bounding reactor; t, time; V_r, V_φ, V_z, components of flow velocity vector; ν, kinematic viscosity coefficient; ρ, density; $a = \nu \ln(r_1/r_0)$; $b = (r_1^2 - r_0^2)/2r_1^2r_0^2$; $c = -\Delta p/\rho$; $d = f(h)$; $f(z, t) = rV_r$.

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